**PROBLEM 2**

**Definition** of *factor-square property* (FSP):

A polynomial *f (x)* has the factor-square property (or FSP) if *f (x)* is a factor of *f (x2).*

**Question:**

**What patterns do monic FSP polynomials satisfy?**

p.s. Monic FSP polynomials have 1 as highest-degree coefficient.

**Solution:**

Instead of separately answering the four questions, I tend to solve the **general patterns that monic FSP polynomials satisfy** and then to specialize into monic FSP polynomials of different degrees.

Suppose the roots for *f (x)=0* is ,,…, ,,…*∈ C;*

Set *A ={* ,,…*}*

Obviously, the roots for *f (x2)=0* is , , …, , , …*∈ C .*

Suppose Set *B ={* , , …*}*

According to the definition of FSP polynomials, *f (x)| f (x2)*

i.e. *f (x2) = f (x)· g(x)*

We can get *A ⊆ B*

i.e. ∀x*∈A,* ∃y*∈B* for which *x=y*

Since the roots for *f (x2)=0* are the square roots of *f (x)=0*

Thus, *y2∈A*

(see from its form: *y=*, *y2=x1∈A*)

Suppose sequence {} = , , …*∈A, n∈*

If *∈A*, then , …*∈A, n∈*

If and only if =1, or equals the previous, t∈, the squaring of stops. (seen it as a cyclical group). **[stop condition]**

1. If || >1, then the sequence {} is divergent which contradict with its finite property (infinite number of roots).
2. If || ∈(0,1), then the sequence {} is an infinite sequence which contradict with its finite property.
3. Derivate from above, ||=0, or ||=1

Thus, the expression for *f(x)* is:

***f(x) = [(x-) (x-)…(x-)] [(x-) (x-)…(x-)] [(x-) (x-)…(x-)],***

where *, , … , n∈*

Now, I specialize the general pattern shown above into monic FSP polynomials of **different degrees**.

**Monic FSP polynomials of degree 1:**

**Guiding Question (a)**:

Are *x* and *x − 1* the only monic FSP polynomials of degree 1?

**Solutions:**

According to the stop condition, there are two possible situations:

Solving the equation, we get: .

Thus, or

i.e. *x* and *x − 1* are the only monic FSP polynomials of degree 1.

**Monic FSP polynomials of degree 2:**

**Guiding Question (b):**

List all the monic FSP polynomials of degree 2.

To start, note that *x2*, *x2 − 1*, *x2 − x*, and *x2 + x + 1* are on that list. Some of them are products

of FSP polynomials of smaller degree. For instance, *x2* and *x2 − x* arise from degree 1 cases. However, *x2 − 1* and *x2 + x + 1* are new, not expressible as a product of two smaller FSP polynomials.

Which terms in your list of degree 2 examples are new?

**Solutions:**

Solving the equation, we get: or

or

Solving the equation, we get:, or, or

or or

Solving the equation, we get:, or, or

or or

To sum up under this situation,

According to the stop condition,

Solving the equation set above, we get:

or or or

To sum up under this situation,

Thus, **all the monic FSP polynomials of degree 2 are:**

Among them, all the FSP polynomials calculated from ii) (i.e. ) arise from degree 1 cases.

Thus, except those,

are new FSP polynomials.

**Monic FSP polynomials of degree 3:**

**Guiding Question (c1):**

List all the monic FSP polynomials of degree 3. Which of those are new?

**Solutions:**

Similar to FSP polynomials of degree 2:

According to the stop condition, we get:

According to the stop condition, we get:

According to the stop condition, we get:

Thus, **all the monic FSP polynomials of degree 3 are listed above (with repetition),** and **in situation ii) and iii), the FSP polynomials of degree 3 are expressible as a product of several smaller FSP polynomials:**

which means that **expect those, are new.**

**Guiding Question (c2):**

List the monic FSP polynomials of degree 3 that have integer coefficients; Separately list those (if any) with complex number coefficients that are not all integers.

**Solutions:**

Monic FSP polynomials of degree 3 that have integer coefficients:

Monic FSP polynomials of degree 3 with complex number coefficients that are not all integers:

**Monic FSP polynomials of degree 4:**

**Guiding Question (d):**

List all the monic FSP polynomials of degree 4. Which of those are new?

List the monic FSP polynomials of degree 4 that have integer coefficients; Separately list those (if any) with complex number coefficients that are not all integers.

**Solutions:**

Similar to FSP polynomials of degree 3:

According to the stop condition, we get:

According to the stop condition, we get:

while

or

According to the stop condition, we get:

, or, or

while

, or, or

According to the stop condition, we get:

, or, or

while

or and or

According to the stop condition, we get:

or ,

For those five situations, different (t∈Ζ+) are calculated based on the stop condition; and **in situation ii), iii), iv), and v), the FSP polynomials of degree 4 are expressible as a product of several smaller FSP polynomials,** which means that **expect those, are new.**

Monic FSP polynomials of degree 4 with complex number coefficients that are not all integers:

*where*

*while*

or

Except those, are Monic FSP polynomials of degree 4 that have integer coefficients.

**Moreover,** according to the stop condition, the coefficients can only be “±1” or complex numbers; thus, there aren’t examples of monic FSP polynomials with real number coefficients that are not all integers.